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## **Construction of Three Dimensional Solutions for the Maxwell Equations**

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# CONSTRUCTION OF THREE DIMENSIONAL SOLUTIONS FOR THE MAXWELL EQUATIONS\*

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**Abstract.** We consider numerical solutions for the three dimensional time dependent Maxwell equations. We construct a fourth order accurate compact implicit scheme and compare it to the Yee scheme for free space in a box.

**Subject classification.** Applied and Numerical Mathematics

**Key words.** Maxwell equations, the Yee scheme, the Ty(2,4) scheme

**1. Maxwell Equations in a Box.** Let  $\tau = ct = t/\sqrt{\mu\epsilon}$  and  $Z = \sqrt{\frac{\mu}{\epsilon}}$ . For the rest of this paper we replace  $\tau$  by  $t$ . The three dimensional time dependent Maxwell equations then are:

$$\begin{aligned}
 \frac{\partial E_x}{\partial t} &= Z \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \\
 \frac{\partial E_y}{\partial t} &= Z \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \\
 \frac{\partial E_z}{\partial t} &= Z \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \\
 \frac{\partial H_x}{\partial t} &= \frac{1}{Z} \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) \\
 \frac{\partial H_y}{\partial t} &= \frac{1}{Z} \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \\
 \frac{\partial H_z}{\partial t} &= \frac{1}{Z} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right).
 \end{aligned}
 \tag{1.1}$$

We set  $Z = 1$  in this paper.

A plane wave solution is given by

$$\begin{aligned}
 H_x &= H_x^0 \sin(\omega t) \sin(Ax + By + Cz) \\
 H_y &= H_y^0 \sin(\omega t) \sin(Ax + By + Cz) \\
 H_z &= H_z^0 \sin(\omega t) \sin(Ax + By + Cz) \\
 E_x &= E_x^0 \cos(\omega t) \cos(Ax + By + Cz) \\
 E_y &= E_y^0 \cos(\omega t) \cos(Ax + By + Cz) \\
 E_z &= E_z^0 \cos(\omega t) \cos(Ax + By + Cz)
 \end{aligned}$$

Substituting into the Maxwell equations this is a solution if

$$\omega^2 = A^2 + B^2 + C^2
 \tag{1.2}$$

$$0 = AH_x^0 + BH_y^0 + CH_z^0
 \tag{1.3}$$

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We also demand that

$$\begin{aligned}\omega E_x^0 &= H_y^0 C - H_z^0 B \\ \omega E_y^0 &= H_z^0 A - H_x^0 C \\ \omega E_z^0 &= H_x^0 B - H_y^0 A\end{aligned}$$

**2. Numerical Tests.** We consider a case where  $H_x^0 = H_y^0 = H_z^0 = 1$  and

$$\begin{aligned}A &= \pi \\ B &= -2\pi \\ C &= \pi \\ \omega &= \sqrt{6}\pi\end{aligned}$$

We use this exact solution as a basis for comparison in the box  $[0, 1/2] \times [0, 1/4] \times [0, 1/2]$ . We shall compare two numerical methods: the Yee scheme [1] which is second order accurate in space and time and the Ty(2,4) scheme [2, 3] which is second order accurate in time but fourth order accurate in space. In order for the total error to be fourth order we must choose the time step small enough so that the temporal error does not swamp the spatial error. This requires  $\Delta t \sim (\Delta x)^2$ . If the error requirements are too severe then this is inefficient and the leapfrog in time should be replaced by a fourth order Runge-Kutta method. However, for the experiments in this paper we shall use the same leapfrog method for both schemes. Hence, both the Yee scheme and the Ty(2,4) have the electric and magnetic variables at the same staggered locations both in space and in time. The Yee scheme approximates the derivatives via the following approximation.

$$\frac{\partial}{\partial y} \begin{bmatrix} U^{1/2} \\ U^{3/2} \\ \cdot \\ \cdot \\ U^{(2p-1)/2} \end{bmatrix} = \frac{1}{\Delta y} \left( \begin{bmatrix} U^1 \\ U^2 \\ \cdot \\ U^{p-1} \\ U^p \end{bmatrix} - \begin{bmatrix} U^0 \\ U^1 \\ \cdot \\ U^{p-2} \\ U^{p-1} \end{bmatrix} \right).$$

A similar formula holds for the other variables shifted to other locations in each direction. The Ty(2,4) scheme is an implicit compact scheme given by

$$\frac{\partial}{\partial y} \begin{bmatrix} U^{1/2} \\ U^{3/2} \\ \cdot \\ \cdot \\ U^{(2p-1)/2} \end{bmatrix} = \mathbf{A}^{-1} \frac{1}{\Delta y} \left( \begin{bmatrix} U^1 \\ U^2 \\ \cdot \\ U^{p-1} \\ U^p \end{bmatrix} - \begin{bmatrix} U^0 \\ U^1 \\ \cdot \\ U^{p-2} \\ U^{p-1} \end{bmatrix} \right).$$

where  $\mathbf{A}$  is defined the following way:

$$\mathbf{A} = \frac{1}{24} \begin{pmatrix} 26 & -5 & 4 & -1 & \cdot & \cdot & 0 \\ 1 & 22 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & 22 & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \\ 0 & \cdot & \cdot & 0 & 1 & 22 & 1 \\ 0 & \cdot & \cdot & -1 & 4 & -5 & 26 \end{pmatrix}.$$

For the Yee scheme we choose  $\Delta t = \frac{4h}{7}$  while for the Ty(2,4) scheme we choose  $\Delta t \sim h^2$  where  $h = \Delta x = \Delta y$ .

We measure the error in the  $L_2$  norm between the approximate and exact electric field in the  $\hat{z}$ -direction. The Ty(2,4) behaves better than expected and gives almost fifth order accuracy. The Yee scheme gives a second order accuracy as expected.

## REFERENCES

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<i>scheme</i>	<i>h</i>	$\Delta t$	t=10	reduction	rate
<i>Ty</i> (2, 4)	$\frac{1}{20}$	$\frac{1}{400}$	$3.62 \times 10^{-4}$		
<i>Ty</i> (2, 4)	$\frac{1}{40}$	$\frac{1}{1600}$	$1.1443 \times 10^{-5}$	31.6423	4.98
<i>Ty</i> (2, 4)	$\frac{1}{80}$	$\frac{1}{6400}$	$3.5621 \times 10^{-7}$	32.1255	5.0056
<i>Yee</i>	$\frac{1}{20}$	$\frac{1}{35}$	0.027		
<i>Yee</i>	$\frac{1}{40}$	$\frac{1}{70}$	$7.3 \times 10^{-4}$	3.694	1.9028
<i>Yee</i>	$\frac{1}{80}$	$\frac{1}{140}$	$1.82 \times 10^{-4}$	4.0042	2.0015

TABLE 2.1  
Comparison of the maximum errors in  $L_2$  norm

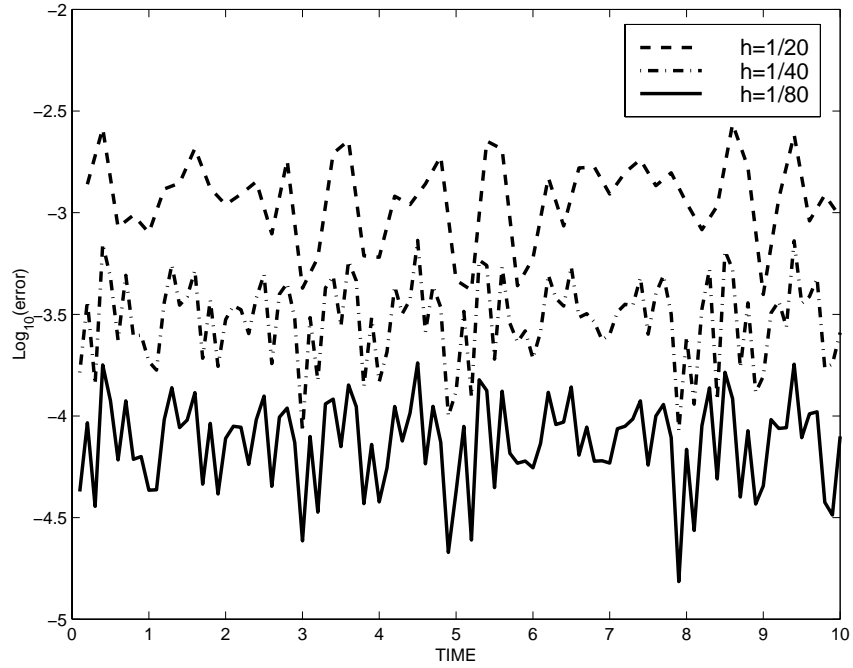


FIG. 2.1.  $\log_{10}(\text{error})$  For the Yee scheme.

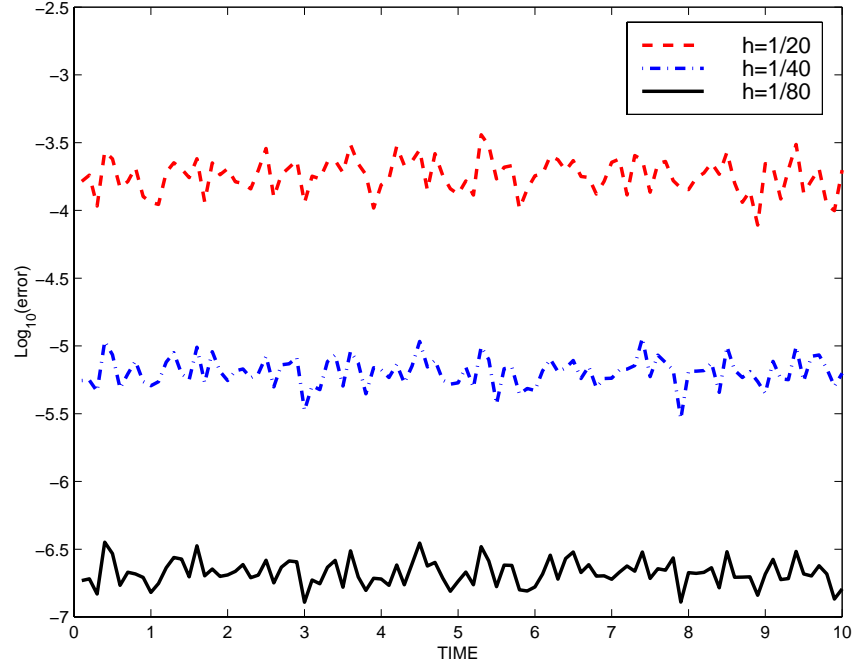


FIG. 2.2.  $\log_{10}(\text{errors})$  For the  $Ty(2,4)$  scheme.

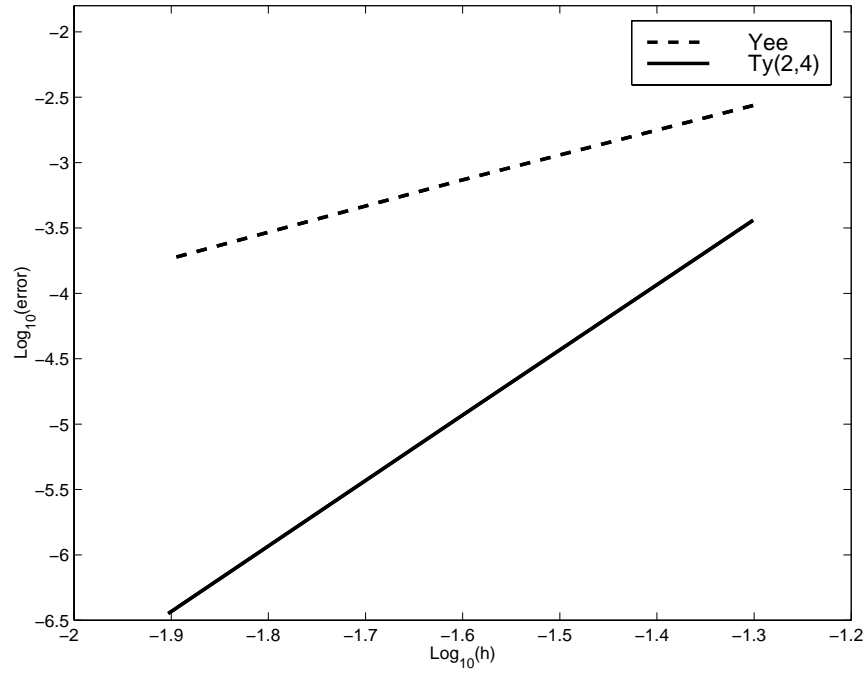


FIG. 2.3.  $\log_{10}(\text{error})$  as a function of  $\log_{10}(h)$  For the Yee and the  $Ty(2,4)$  schemes.